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NONSTATIONARY HEAT TRANSFER IN A HOLLOW COMPOSITE CYLINDER

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Solution of a nonstationary heat-transfer problem for bounded end-conjugated hollow dissimilar cylinders is presented. In the volume of the cylinders, time- and coordinate-dependent heat release of known intensity takes place. The problem is solved using finite integral transformations over two coordinates.

Key words: *heat conduction equation, composite cylinder, finite integral transformation, boundary conditions.*

As opposed to the well-known stationary problem [1], here we treat a nonstationary heat-conduction problem for a hollow composite cylinder with first-, second-, or third-kind heat-transfer conditions at the external boundaries dependent both on time and coordinates. At the junction interface of the composite cylinder, an ideal thermal contact (fourth-kind boundary conditions) is assumed. Next, we assume that in the volume of the constituent cylinders time- and coordinate-dependent heat release takes place. The cylindrical coordinate system used in the present consideration and the dimensions of the composite cylinder are shown in Fig. 1. The problem on determination of the temperature field in the composite cylinder can be represented in the form of two heat conduction equations, conditions posed at the external boundaries and at the interface between the cylinders, and initial conditions.

Heat conduction equation:

$$\frac{1}{\chi_1}\frac{\partial T_1}{\partial \tau} = \frac{\partial^2 T_1}{\partial r^2} + \frac{1}{r}\frac{\partial T_1}{\partial r} + \frac{\partial^2 T_1}{\partial z^2} + \frac{1}{\lambda_1} w_1, \qquad a \leqslant r \leqslant b, \quad c \leqslant z \leqslant d,$$
$$T_1 = T_1(\tau, r, z); \tag{1}$$

$$\frac{1}{\chi_2}\frac{\partial T_2}{\partial \tau} = \frac{\partial^2 T_2}{\partial r^2} + \frac{1}{r}\frac{\partial T_2}{\partial r} + \frac{\partial^2 T_2}{\partial z^2} + \frac{1}{\lambda_2} w_2, \qquad a \leqslant r \leqslant b, \quad d \leqslant z \leqslant e,$$
$$T_2 = T_2(\tau, r, z); \tag{2}$$

boundary conditions (first-kind conditions):

— on the cylindrical surfaces,

$$T_1\Big|_{r=a} = T_{1a}(\tau, z), \quad T_1\Big|_{r=b} = T_{1b}(\tau, z), \quad T_2\Big|_{r=a} = T_{2a}(\tau, z), \quad T_2\Big|_{r=b} = T_{2b}(\tau, z); \tag{3}$$

— on the end surfaces of the cylinder,

$$T_1\Big|_{z=c} = T_c(\tau, r), \quad T_1\Big|_{z=d} = T_2\Big|_{z=d}, \quad \lambda_1\frac{\partial T_1}{\partial z}\Big|_{z=d} = \lambda_2\frac{\partial T_2}{\partial z}\Big|_{z=d}, \quad T_2\Big|_{z=e} = T_e(\tau, r); \tag{4}$$

initial conditions:

$$T_1\Big|_{\tau=0} = T_{10}(r, z), \qquad T_2\Big|_{\tau=0} = T_{20}(r, z). \tag{5}$$

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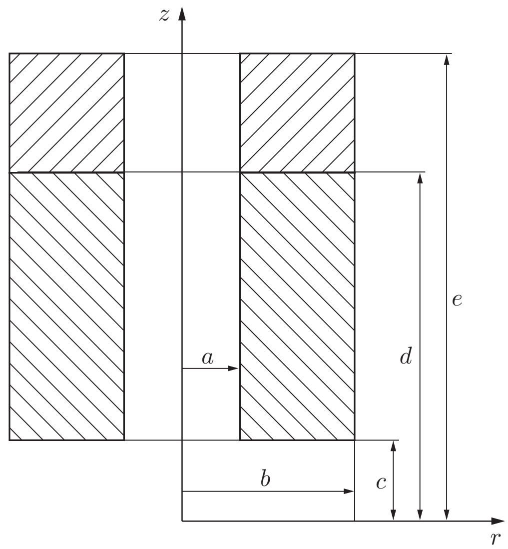

Fig. 1. Geometry of the composite cylinder.

Here T_1 and T_2 are the temperatures in the first and second cylinder, respectively, χ_1 and χ_2 are thermal diffusivities, λ_1 and λ_2 are the heat-transfer coefficients, w_1 and w_2 are the rates of heat generation in the volumes of the cylinders, τ is the time, r and z are the coordinates of the cylinders (radius and height), and a, b, c, d, and e are the geometric characteristics of the cylinders (see Fig. 1).

We apply the following change of variables:

$$\begin{aligned} &z = \sqrt{\chi_1}\, y, \qquad r = \sqrt{\chi_1}\, x \qquad \text{at} \quad c \leqslant z \leqslant d, \quad a \leqslant r \leqslant b, \\ &z = \sqrt{\chi_2}\, y, \qquad r = \sqrt{\chi_2}\, x \qquad \text{for} \quad d \leqslant z \leqslant e, \quad a \leqslant r \leqslant b. \end{aligned} \tag{6}$$

Then, relations (1)–(5) acquire the form

$$\begin{gathered} \frac{\partial Z_1}{\partial \tau} = \frac{\partial^2 Z_1}{\partial x^2} + \frac{1}{x}\frac{\partial Z_1}{\partial x} + \frac{\partial^2 Z_1}{\partial y^2} + \frac{\chi_1}{\lambda_1} w_{11}, \qquad x_1 \leqslant x \leqslant x_2, \quad y_1 \leqslant y \leqslant y_2, \\ Z_1(\tau, x, y) = T_1(\tau, r, z); \end{gathered} \tag{7}$$

$$\begin{gathered} \frac{\partial Z_2}{\partial \tau} = \frac{\partial^2 Z_2}{\partial x^2} + \frac{1}{x}\frac{\partial Z_2}{\partial x} + \frac{\partial^2 Z_2}{\partial y^2} + \frac{\chi_2}{\lambda_2} w_{22}, \qquad x_3 \leqslant x \leqslant x_4, \quad y_3 \leqslant y \leqslant y_4, \\ Z_2(\tau, x, y) = T_2(\tau, r, z); \end{gathered} \tag{8}$$

$$Z_1\Big|_{x=x_1} = T_{1a}(\tau, y), \quad Z_1\Big|_{x=x_2} = T_{1b}(\tau, y), \quad Z_2\Big|_{x=x_3} = T_{2a}(\tau, y), \quad Z_2\Big|_{x=x_4} = T_{2b}(\tau, y); \tag{9}$$

$$\begin{gathered} Z_1\Big|_{y=y_1} = T_c(\tau, x), \qquad Z_1\Big|_{y=y_2} = Z_2\Big|_{y=y_3}, \\ b_1 \frac{\partial Z_1}{\partial y}\Big|_{y=y_2} = b_2 \frac{\partial Z_2}{\partial y}\Big|_{y=y_3}, \qquad Z_2\Big|_{y=y_4} = T_e(\tau, x); \end{gathered} \tag{10}$$

$$Z_1\Big|_{\tau=0} = T_{10}(x, y), \qquad Z_2\Big|_{\tau=0} = T_{20}(x, y). \tag{11}$$

Here $y_1 = c/\sqrt{\chi_1}$, $y_2 = d/\sqrt{\chi_1}$, $y_3 = d/\sqrt{\chi_2}$, $y_4 = e/\sqrt{\chi_2}$, $x_1 = a/\sqrt{\chi_1}$, $x_2 = b/\sqrt{\chi_1}$, $x_3 = a/\sqrt{\chi_2}$, $x_4 = b/\sqrt{\chi_2}$, $b_1 = \lambda_1/\sqrt{\chi_1}$, $b_2 = \lambda_2/\sqrt{\chi_1}$, $w_{11}(\tau, x, y) = w_1(\tau, r, z)$, and $w_{22}(\tau, x, y) = w_2(\tau, r, z)$.

To solve system (7)–(11), we define the integral transformation

$$\overline{Z}(\tau,x,s) = A_1 \int\limits_{y_1}^{y_2} Z_1(\tau,x,y)U_1(sy)\,dy + A_2 \int\limits_{y_3}^{y_4} Z_2(\tau,x,y)U_2(sy)\,dy, \tag{12}$$

whose functions $U_1(sy)$ and $U_2(sy)$ satisfy the following equations and boundary conditions:

$$\frac{d^2U_1}{dy^2} + s^2U_1 = 0 \quad (y_1 \leqslant y \leqslant y_2), \qquad \frac{d^2U_2}{dy^2} + s^2U_2 = 0 \quad (y_3 \leqslant y \leqslant y_4); \tag{13}$$

$$U_1\Big|_{y_1} = 0, \qquad U_1\Big|_{y_2} = U_2\Big|_{y_3}, \qquad b_1\,\frac{dU_1}{dy}\Big|_{y_2} = b_2\,\frac{dU_2}{dy}\Big|_{y_3}, \qquad U_2\Big|_{y_4} = 0. \tag{14}$$

The solutions of (13) are

$$U_1(sy) = C_1\sin sy + C_2\cos sy, \qquad U_2(sy) = C_3\sin sy + C_4\cos sy,$$

where C_1, C_2, C_3, and C_4 are arbitrary constants and s are characteristic numbers.

Conditions (14) yield the following system of equations

$$\begin{gathered} C_1\sin sy_1 + C_2\cos sy_1 = 0,\\ C_1\sin sy_2 + C_2\cos sy_2 - C_3\sin sy_3 - C_4\cos sy_3 = 0,\\ b_1(C_1\cos sy_2 - C_2\sin sy_2) - b_2(C_3\cos sy_3 - C_2\sin sy_3) = 0,\\ C_3\sin sy_4 + C_4\cos sy_4 = 0. \end{gathered} \tag{15}$$

System (15) has a nontrivial solution iff the determinant of the system is zero. The latter condition yields the following equation for the characteristic numbers s:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0. \tag{16}$$

Here $a_{11} = \sin sy_1$, $a_{12} = \cos sy_1$, $a_{21} = \sin sy_2$, $a_{22} = \cos sy_2$, $a_{23} = -\sin sy_3$, $a_{24} = -\cos sy_3$, $a_{31} = b_1\cos sy_2$, $a_{32} = -b_1\sin sy_2$, $a_{33} = -b_2\cos sy_3$, $a_{32} = b_2\sin sy_3$, $a_{43} = \sin sy_4$, and $a_{44} = \cos sy_4$.

By means of rather simple transformations we eliminate the indefinite constants C_2, C_3, and C_4 and obtain the following expressions for the functions $U_1(sy)$ and $U_2(sy)$:

$$U_1(sy) = C_1(\sin sy + f_1\cos sy), \qquad U_2(sy) = C_1(f_3\sin sy + f_4\cos sy). \tag{17}$$

The coefficients f_i $(i = 1,3,4)$ contain the quantities $a_{11}, a_{12}, a_{21}, \ldots$.

Integral transformation (12) and expressions (17) still contain indefinite constants A_1, A_2, and C_1. The values of A_1 and A_2 can be found from the orthogonality condition for the functions $U_1(sy)$ and $U_2(sy)$, and the constant C_1, from the condition of orthonormality of these functions.

Orthogonality of the functions implies that $\int\limits_{y_1}^{y_2} U_1(sy)U_1(py)\,dy = 0$ and $\int\limits_{y_3}^{y_4} U_2(sy)U_2(py)\,dy = 0$ for $s \neq p$; these relations follow from the equality $J = A_1 \int\limits_{y_1}^{y_2} U_1(sy)U_1(py)\,dy + A_2 \int\limits_{y_3}^{y_4} U_2(sy)U_2(py)\,dy = 0$ and from boundary conditions (14) for $s \neq p$, $A_1 = 1/b_2$, and $A_2 = 1/b_1$. Let us prove this statement.

Using Eqs. (13), we can write the quantity J as

$$J = -\frac{A_1}{p^2}\int\limits_{y_1}^{y_2} U_1(sy)\,\frac{d^2U_1}{dy^2}\,dy - \frac{A_2}{p^2}\int\limits_{y_3}^{y_4} U_2(sy)\,\frac{d^2U_2}{dy^2}\,dy.$$

We integrate each of the two integrals by parts; this yields

$$J = -\frac{A_1}{p^2}\Big[U_1(sy)\,\frac{dU_1(py)}{dy}\Big|_{y_1}^{y_2} - \frac{dU_1(sy)}{dy}\,U_1(py)\Big|_{y_1}^{y_2} + \int\limits_{y_1}^{y_2}\frac{d^2U_1(sy)}{dy^2}\,U_1(py)\,dy\Big]$$

$$-\frac{A_2}{p^2}\Big[U_2(sy)\,\frac{dU_2(py)}{dy}\Big|_{y_3}^{y_4} - \frac{dU_2(sy)}{dy}\,U_2(py)\Big|_{y_3}^{y_4} + \int\limits_{y_3}^{y_4}\frac{d^2U_2(sy)}{dy^2}\,U_2(py)\,dy\Big].$$

We take conditions (14) into account and assume that $A_1 = 1/b_2$ and $A_2 = 1/b_1$; then, we obtain

$$J = \frac{s^2}{b_2p^2}\int\limits_{y_1}^{y_2} U_1(sy)U_1(py)\,dy + \frac{s^2}{b_1p^2}\int\limits_{y_3}^{y_4} U_2(sy)U_2(py)\,dy.$$

We compare the obtained expression for J with the initial expression

$$\frac{A_1}{p^2}\int\limits_{y_1}^{y_2} U_1(sy)U_1(py)\,dy + \frac{A}{p^2}\int\limits_{y_3}^{y_4} U_2(sy)U_2(py)\,dy = \frac{s^2}{b_2p^2}\int\limits_{y_1}^{y_2} U_1(sy)U_1(py)\,dy + \frac{s^2}{b_1p^2}\int\limits_{y_3}^{y_4} U_2(sy)U_2(py)\,dy,$$

and arrive at the conclusion that the last identity holds for $s \neq p$ iff

$$\int\limits_{y_1}^{y_2} U_1(sy)U_1(py)\,dy = 0, \qquad \int\limits_{y_3}^{y_4} U_2(sy)U_2(py)\,dy = 0.$$

Thus, orthogonality of $U_1(sy)$ and $U_2(sy)$ is proved and the values of A_1 and A_2 are found. We define the coefficient C_1 so that to make the indicated functions orthonormal:

$$A_1\int\limits_{y_1}^{y_2}[U_1(sy)]^2\,dy + A_2\int\limits_{y_3}^{y_4}[U_2(sy)]^2\,dy = 1$$

or

$$A_1\int\limits_{y_1}^{y_2}[C_1(\sin sy + f_1\cos sy)]^2\,dy + A_2\int\limits_{y_3}^{y_4}[C_1(f_3\sin sy + f_4\cos sy)]^2\,dy = 1.$$

It follows from this condition that $C_1 = 1/\sqrt{I}$, where

$$I = A_1\Big[(1+f_1^2)\,\frac{y_2-y_1}{2} + \frac{f_1^2-1}{4s}\,(\sin 2sy_2 - \sin 2sy_1) - \frac{f_1}{2s}\,(\cos 2sy_2 - \cos 2sy_1)\Big]$$

$$+ A_2\Big[(f_3^2+f_4^2)\,\frac{y_4-y_3}{2} + \frac{f_4^2-f_3^2}{4s}\,(\sin 2sy_4 - \sin 2sy_3) - \frac{f_3f_4}{2s}\,(\cos 2sy_4 - \cos 2sy_3)\Big].$$

Thus, integral transformation (12) is defined.

We apply the integral transformation to Eqs. (7) and (8); this yields

$$A_1\int\limits_{y_1}^{y_2}\Big(\frac{\partial Z_1}{\partial\tau} - \frac{\partial^2 Z_1}{\partial x^2} - \frac{1}{x}\,\frac{\partial Z_1}{\partial x} - \frac{\partial^2 Z_1}{\partial y^2} - \frac{\chi_1}{\lambda_1}\,w_{11}\Big)U_1(sy)\,dy$$

$$+ A_2\int\limits_{y_3}^{y_4}\Big(\frac{\partial Z_2}{\partial\tau} - \frac{\partial^2 Z_2}{\partial x^2} - \frac{1}{x}\,\frac{\partial Z_2}{\partial x} - \frac{\partial^2 Z_2}{\partial y^2} - \frac{\chi_2}{\lambda_2}\,w_{22}\Big)U_2(sy)\,dy = 0. \qquad (18)$$

As a result, relation (18) assumes the form

$$\frac{\partial\overline{Z}}{\partial\tau} = \frac{\partial^2\overline{Z}}{\partial x^2} + \frac{1}{x}\,\frac{\partial\overline{Z}}{\partial x} - s^2\overline{Z} + FP_1(\tau,x,s) + FP_2(\tau,x,s), \qquad (19)$$

where

$$FP_1(\tau, x, s) = A_1 \, \frac{dU_1}{dy}\Big|_{y_1} T_c(\tau, x) - A_2 \, \frac{dU_2}{dy}\Big|_{y_4} T_e(\tau, x); \tag{20}$$

$$FP_2(\tau, x, s) = A_1 \, \frac{\chi_1}{\lambda_1} \int\limits_{y_1}^{y_2} w_{11} U_1(sy)\, dy + A_2 \, \frac{\chi_2}{\lambda_2} \int\limits_{y_3}^{y_4} w_{22} U_2(sy)\, dy. \tag{21}$$

By definition of integral transformation (12), Eq. (19) is valid in the intervals $x_1 \leqslant x \leqslant x_2$ and $x_3 \leqslant x \leqslant x_4$. Let us find the solution of (19) in each of these intervals. To this end, we represent the conditions at the boundaries x_1, x_2, x_3, and x_4 as

$$\overline{Z}_1\Big|_{x=x_1} = \int\limits_{y_1}^{y_2} U_1(sy) T_{1a}(\tau, y)\, dy, \qquad \overline{Z}_1\Big|_{x=x_2} = \int\limits_{y_1}^{y_2} U_1(sy) T_{1b}(\tau, y)\, dy,$$

$$\overline{Z}_2\Big|_{x=x_3} = \int\limits_{y_3}^{y_4} U_2(sy) T_{2a}(\tau, y)\, dy, \qquad \overline{Z}_2\Big|_{x=x_4} = \int\limits_{y_3}^{y_4} U_2(sy) T_{2b}(\tau, y)\, dy.$$

To solve the equation in the intervals $x_1 \leqslant x \leqslant x_2$ and $x_3 \leqslant x \leqslant x_4$, we define the integral transformations

$$\overline{\overline{Z}}_1(\tau, p, s) = \int\limits_{x_1}^{x_2} x\, \overline{Z}(\tau, x, s)\, V_1(px)\, dx, \qquad \overline{\overline{Z}}_2(\tau, q, s) = \int\limits_{x_3}^{x_4} x\, \overline{Z}(\tau, x, s)\, V_2(qx)\, dx, \tag{22}$$

where $V_1(px)$ and $V_2(qx)$ are the solutions of the differential equations

$$\frac{d^2V_1}{dx^2} + \frac{1}{x}\,\frac{dV_1}{dx} + p^2 V_1 = 0, \qquad \frac{d^2V_2}{dx^2} + \frac{1}{x}\,\frac{dV_2}{dx} + q^2 V_2 = 0 \tag{23}$$

with the boundary conditions

$$V_1\Big|_{x_1} = 0, \qquad V_1\Big|_{x_2} = 0, \qquad V_2\Big|_{x_3} = 0, \qquad V_2\Big|_{x_4} = 0. \tag{24}$$

The solutions of (23) have the form [2]

$$V_1(px) = D_1 J_0(px) + D_2 Y_0(px), \qquad V_2(qx) = D_3 J_0(qx) + D_4 Y_0(qx),$$

where D_1, D_2, D_3, and D_4 are arbitrary constants; $J_0(z)$ and $Y_0(z)$ are the zero-order Bessel functions of the first and second kind; and p and q are the characteristic numbers.

Using the boundary conditions (24), we obtain the following two systems of equations:

$$\begin{aligned} &D_1 J_0(px_1) + D_2 Y_0(px_1) = 0, \qquad D_1 J_0(px_2) + D_2 Y_0(px_2) = 0; \\ &D_3 J_0(qx_3) + D_4 Y_0(qx_3) = 0. \qquad D_3 J_0(qx_4) + D_4 Y_0(qx_4) = 0 \end{aligned} \tag{25}$$

These systems have nontrivial solutions iff their determinants are zero. This yields two equations for the characteristic numbers p and q:

$$\begin{vmatrix} J_0(px_1) & Y_0(px_1) \\ J_0(px_2) & Y_0(px_2) \end{vmatrix} = 0, \qquad \begin{vmatrix} J_0(qx_3) & Y_0(qx_3) \\ J_0(qx_4) & Y_0(qx_4) \end{vmatrix} = 0. \tag{26}$$

We eliminate the coefficients D_2 and D_4 out of the systems of (25) and obtain the following expressions for $V_1(px)$ and $V_2(qx)$:

$$V_1(px) = D_1[J_0(px) + f_5 Y_0(px)], \qquad V_2(qx) = D_3[J_0(qx) + f_6 Y_0(qx)], \tag{27}$$

where $f_5 = -J_0(px_1)/Y_0(px_1)$ and $f_6 = -J_0(qx_3)/Y_0(qx_3)$.

It should be noted that solutions (27) with boundary conditions (24) are orthogonal systems of functions $V_1(px)$ and $V_2(qx)$. We demand that these systems be orthonormal, i.e.,

$$\int_{x_1}^{x_2} x\,[V_1(px)]^2\,dx = 1, \qquad \int_{x_3}^{x_4} x\,[V_2(qx)]^2\,dx = 1;$$

this yields the constants D_1 and D_2. Then,

$$D_1 = 1/\sqrt{I_1}, \qquad D_2 = 1/\sqrt{I_2},$$

where

$$I_1 = x_2^2[J_0(px_2)+f_5Y_0(px_2)]^2/2+x_2^2[J_1(px_2)+f_5Y_1(px_2)]^2/2-x_1^2[J_0(px_1)+f_5Y_0(px_1)]^2/2-x_1^2[J_1(px_1)+f_5Y_1(px_1)]^2/2;$$

$$I_2 = x_4^2[J_0(qx_4)+f_6Y_0(qx_4)]^2/2+x_4^2[J_1(qx_4)+f_6Y_1(qx_4)]^2/2-x_3^2[J_0(qx_3)+f_6Y_0(qx_3)]^2/2-x_3^2[J_1(qx_3)+f_6Y_1(qx_3)]^2/2;$$

and J_1 and Y_1 are the first-order Bessel functions of the first and second kind, respectively. Thus, integral transformations (22) are defined.

We apply the obtained integral transformations to relations (19)–(21):

$$\int_{x_1}^{x_2} x\Big[\frac{d\overline{Z}_1}{d\tau} - \frac{\partial^2\overline{Z}_1}{\partial x^2} - \frac{1}{x}\,\frac{\partial\overline{Z}_1}{\partial x} + s^2\overline{Z}_1 - FP_1(\tau,x,s) - FP_2(\tau,x,s)\Big]V_1(px)\,dx = 0,$$

$$\int_{x_3}^{x_4} x\Big[\frac{d\overline{Z}_2}{d\tau} - \frac{\partial^2\overline{Z}_2}{\partial x^2} - \frac{1}{x}\,\frac{\partial\overline{Z}_2}{\partial x} + s^2\overline{Z}_2 - FP_1(\tau,x,s) - FP_2(\tau,x,s)\Big]V_2(qx)\,dx = 0.$$

This yields

$$\begin{aligned}
&\frac{\partial\overline{\overline{Z}}_1}{\partial\tau} - \int_{x_1}^{x_2} x\Big[\frac{\partial^2\overline{Z}_1}{\partial x^2} + \frac{1}{x}\,\frac{\partial\overline{Z}_1}{\partial x}\Big]V_1(px)\,dx + s^2\,\overline{\overline{Z}}_1 - FP_3(\tau,p,s) = 0,\\
&\frac{\partial\overline{\overline{Z}}_2}{\partial\tau} - \int_{x_3}^{x_4} x\Big[\frac{\partial^2\overline{Z}_2}{\partial x^2} + \frac{1}{x}\,\frac{\partial\overline{Z}_2}{\partial x}\Big]V_2(qx)\,dx + s^2\,\overline{\overline{Z}}_2 - FQ_3(\tau,q,s) = 0,
\end{aligned} \qquad (28)$$

where

$$FP_3(\tau,p,s) = \int_{x_1}^{x_2} x\,[FP_1(\tau,x,s)+FP_2(\tau,x,s)]V_1(px)\,dx; \quad FQ_3(\tau,q,s) = \int_{x_3}^{x_4} x\,[FP_1(\tau,x,s)+FP_2(\tau,x,s)]V_2(qx)\,dx.$$

After repeated integration by parts, the differential operators under the sign of integration in (28) transform into

$$\int_{x_1}^{x_2} x\Big[\frac{\partial^2\overline{Z}_1}{\partial x^2} + \frac{1}{x}\,\frac{\partial\overline{Z}_1}{\partial x}\Big]\;V_1(px)\,dx = FP_4(\tau,p,s) - p^2\,\overline{\overline{Z}}_1,$$

$$\int_{x_3}^{x_4} x\Big[\frac{\partial^2\overline{Z}_2}{\partial x^2} + \frac{1}{x}\,\frac{\partial\overline{Z}_2}{\partial x}\Big]V_2(qx)\,dx = FQ_4(\tau,q,s) - q^2\,\overline{\overline{Z}}_2.$$

Equations (28) acquire the form

$$\frac{d\overline{\overline{Z}}_1}{d\tau} + (s^2+p^2)\,\overline{\overline{Z}}_1 = FP_3(\tau,p,s) + FP_4(\tau,p,s), \qquad \frac{d\overline{\overline{Z}}_2}{d\tau} + (s^2+q^2)\,\overline{\overline{Z}}_2 = FQ_3(\tau,q,s) + FQ_4(\tau,q,s), \quad (29)$$

where

$$FP_4(\tau,p,s) = x_1\,\frac{dV_1}{dx}\Big|_{x_1}\,\overline{Z}_1\Big|_{x_1} - x_2\,\frac{dV_1}{dx}\Big|_{x_2}\,\overline{Z}_1\Big|_{x_2}, \qquad FQ_4(\tau,q,s) = x_3\,\frac{dV_2}{dx}\Big|_{x_3}\,\overline{Z}_2\Big|_{x_3} - x_4\,\frac{dV_2}{dx}\Big|_{x_4}\,\overline{Z}_2\Big|_{x_4}. \quad (30)$$

The initial conditions for Eqs. (29) can be obtained by applying integral transformations (12) and (22) to relations (11):

$$\overline{\overline{Z}}_1\Big|_{\tau=0} = \int\limits_{x_1}^{x_2} x\Big[A_1 \int\limits_{y_1}^{y_2} T_{01}(x,y)U_1(sy)\,dy + A_2 \int\limits_{y_3}^{y_4} T_{02}(x,y)U_2(sy)\,dy\Big]V_1(px)\,dx,$$

$$\overline{\overline{Z}}_2\Big|_{\tau=0} = \int\limits_{x_3}^{x_4} x\Big[A_1 \int\limits_{y_1}^{y_2} T_{01}(x,y)U_1(sy)\,dy + A_2 \int\limits_{y_3}^{y_4} T_{02}(x,y)U_2(sy)\,dy\Big]V_2(qx)\,dx.$$

The solutions of (29) are

$$\overline{\overline{Z}}_1(\tau,p,s) = \exp\left[-(s^2+p^2)\tau\right]\Big[\int\limits_0^\tau (FP_3(\tau,p,s) + FP_4(\tau,p,s))\exp\left[(s^2+p^2)\tau\right]d\tau + \overline{\overline{Z}}_1\Big|_{\tau=0}\Big],$$

$$\overline{\overline{Z}}_2(\tau,q,s) = \exp\left[-(s^2+q^2)\tau\right]\Big[\int\limits_0^\tau (FQ_3(\tau,q,s) + FQ_4(\tau,q,s))\exp\left[(s^2+q^2)\tau\right]d\tau + \overline{\overline{Z}}_2\Big|_{\tau=0}\Big].$$

Since the functions $U_1(sx)$, $U_2(sx)$, $V_1(py)$, and $V_2(qy)$ are orthonormal, the final solution of the problem is

$$T_1(\tau,r,z) = \sum_{p_i}\Big(\sum_{s_j}\overline{\overline{Z}}_1(\tau,p_i,s_j)U_1(s_jy)\Big)V_1(p_ix), \qquad T_2(\tau,r,z) = \sum_{q_i}\Big(\sum_{s_j}\overline{\overline{Z}}_2(\tau,q_i,s_j)U_2(s_jy)\Big)V_2(q_ix)$$

[the summation here is performed over positive roots p_i, q_i, and s_j of (16) and (26)]. The passage from the coordinates x and y to the coordinates r z can be performed by relations (6).

The obtained solution of the heat conduction problem for composite cylinder with first-kind boundary conditions can easily be extended to cases with other boundary conditions. In particular, on different surfaces of the cylinder heat transfer with different boundary condition is possible. In this case, in addition to boundary conditions (3), (4), (9), and (10) in the initial equations and to boundary conditions (14) and (24), Eqs. (16) and (26) for the characteristic numbers and expressions (20) and (30) will also suffer changes.

Consider, for instance, the following boundary conditions:

— on the bottom and top boundaries of the composite cylinder, second-kind conditions

$$\lambda_1 \frac{\partial T_1}{\partial z}\Big|_{z=c} + g_1(\tau,r) = 0, \qquad \lambda_2 \frac{\partial T_2}{\partial z}\Big|_{z=e} - g_2(\tau,r) = 0;$$

— on the inner surface of the lower cylinder, second-kind condition

$$\lambda_1 \frac{\partial T_1}{\partial r}\Big|_{r=a} + g_3(\tau,z) = 0,$$

— on the outer surface of the lower cylinder, third-kind condition

$$\lambda_1 \frac{\partial T_1}{\partial r}\Big|_{r=b} + \alpha_1[T_1\Big|_{r=b} - T_{1\text{amb}}(\tau,z)] = 0;$$

— on the inner surface of the upper cylinder, first-kind condition

$$T_2\Big|_{r=a} = T_{2a}(\tau,z),$$

— on the outer surface of the upper cylinder, third-type condition

$$\lambda_1 \frac{\partial T_2}{\partial r}\Big|_{r=b} + \alpha_2[T_2\Big|_{r=b} - T_{2\text{amb}}(\tau,z)] = 0.$$

Here g_1, g_2, and g_3 are the rates of heat generation; α_1 and α_2 are the heat-release coefficients on the surfaces of the cylinder, and T_{amb} is the ambient temperature.

Then, the first and fourth equations in (15) acquire the form

$$C_1\cos sy_1 - C_2\sin sy_1 = 0, \qquad C_3\cos sy_4 - C_4\sin sy_4 = 0,$$

and Eq. (25) becomes

$$D_1J_1(px_1) + D_2Y_1(px_1) = 0,$$

$$D_1[-\lambda_1 pJ_1(px_2) + \alpha_1 J_0(px_2)] + D_2[-\lambda_1 pY_1(px_2) + \alpha_1 Y_0(px_2)] = 0,$$

TABLE 1
Temperatures at the Axis of the Solid Cylinder

z, cm	T, °C	
	Method of [1]	Present work
5.0	7.76	7.70
5.6	7.79	7.72
6.2	7.81	7.73
6.8	7.83	7.75
7.4	7.85	7.76
8.0	7.86	7.77
8.6	7.88	7.78
9.2	7.89	7.79
9.8	7.90	7.79
10.4	7.90	7.79
11.0	7.90	7.80

$$D_3 J_0(qx_3) + D_4 Y_0(qx_3) = 0,$$

$$D_3[-\lambda_2 q J_1(qx_4) + \alpha_2 J_0(qx_4)] + D_4[-\lambda_2 q Y_1(qx_4) + \alpha_2 Y_0(qx_4)] = 0.$$

Expressions (20) and (30) can be written as follows:

$$FP_1(\tau, x, s) = A_1 U_1|_{y_1} g_1(\tau, x)/b_1 + A_2 U_2\Big|_{y_4} g_2(\tau, x)/b_2,$$

$$FP_4(\tau, p, s) = x_1 V_1(px_1)\,\overline{g}_3(\tau, s)/\lambda_1 - x_2 V_1(px_2)\alpha_1\,\overline{T}_{1\mathrm{amb}}(\tau, s)/\lambda_1,$$

$$FQ_4(\tau, q, s) = x_3\,\frac{dV_2}{dx}\Big|_{x_3}\overline{T}_{2a}(\tau, s)\Big|_{x_3} + x_4 V_2(qx_4)\,\frac{\alpha_2}{\lambda_2}\,\overline{T}_{2\mathrm{amb}}(\tau, s),$$

where

$$\overline{g}_3(\tau, s) = \int_{y_1}^{y_2} U_1(sy) g_3(\tau, y)\,dy; \qquad \overline{T}_{1\mathrm{amb}}(\tau, s) = \int_{y_1}^{y_2} U_1(sy) T_{1\mathrm{amb}}(\tau, y)\,dy;$$

$$\overline{T}_{2a}(\tau, s) = \int_{y_3}^{y_4} U_2(sy) T_{2a}(\tau, y)\,dy; \qquad \overline{T}_{2\mathrm{amb}}(\tau, s) = \int_{y_3}^{y_4} U_2(sy) T_{2\mathrm{amb}}(\tau, y)\,dy.$$

All other relations remain unchanged.

In numerical calculations, care should be taken to provide for sufficient accuracy in calculating characteristic numbers and in observing the total number of eigenvalues, since both factors critically influence the final accuracy.

We performed a numerical comparison of the obtained analytical solution of the problem about nonstationary heat transfer in a cylinder with the stationary solution of [1]. The temperatures at the axis of a solid cylinder predicted for identical data in the present solution and in the solution of [1] are summarized in Table 1.

Tables 2 and 3 give the temperatures obtained by solving the nonstationary heat-transfer problem for a hollow cylinder with the following boundary conditions:

— on the end surface of the cylinder, first-kind conditions (some surface temperatures T_1 and T_2 are set);

— on the inner cylindrical surfaces, second-kind condition (heat-release intensities g_1 and g_2);

— on the outer cylindrical surfaces, third-kind conditions (heat transfer with the ambient medium with temperatures T_{1c} and T_{2c} and heat-transfer coefficients α_1 and α_2).

The rates of heat release in the volume of the cylinders are w_1 and w_2. The initial temperatures of the cylinders are T_{10} and T_{20}. The initial parameters are as follows: $a = 10$ cm, $b = 15$ cm, $c = 5$ cm, $d = 8$ cm, $e = 11$ cm, $\chi_1 = 4$ cm^2/sec, $\chi_2 = 5$ cm^2/sec, $\lambda_1 = 4.65$ W/(cm·K), $\lambda_2 = 7$ W/(cm·K), $\alpha_1 = 4.6$ W/(cm^2·K), $\alpha_2 = 2.33$ W/(cm^2·K), $g_1 = 1.16$ W/cm^2, $g_2 = 2.32$ W/cm^2, $w_1 = 0.29$ W/cm^3, $w_2 = 0.29$ W/cm^3, $T_{1c} = 1273$ K, $T_{2c} = 1273$ K, $T_1 = 233$ K, $T_2 = 233$ K, $T_{10} = 253$ K, and $T_{20} = 253$ K.

TABLE 2

Temperature Field in the Upper Part of the Composite Cylinder
0.5 h after the Beginning of the Process

z, cm	T, °C					
	r, cm					
	10	11	12	13	14	15
11.0	−40.0	−40.0	−40.0	−40.0	−40.0	−40.0
10.4	151.2	65.4	34.0	21.8	17.8	18.4
9.8	250.4	140.5	86.9	64.3	56.6	56.7
9.2	313.5	193.4	126.0	95.7	85.1	84.6
8.6	355.6	228.3	152.0	116.5	104.1	103.3
8.0	389.4	248.5	165.9	127.6	114.4	114.5

TABLE 3

Temperature Field in the Lower Part of the Hollow Composite Cylinder
0.5 h after the Beginning of the Process

z, cm	T, °C					
	r, cm					
	10	11	12	13	14	15
8.0	389.4	248.5	165.9	127.6	114.4	114.5
7.4	415.04	244.5	157.0	121.9	112.5	115.6
6.8	396.4	225.9	141.5	109.1	101.0	104.7
6.2	335.9	176.0	105.3	79.7	73.9	78.1
5.6	216.3	89.7	47.2	32.9	29.9	33.9
5.0	−40.0	−40.0	−40.0	−40.0	−40.0	−40.0

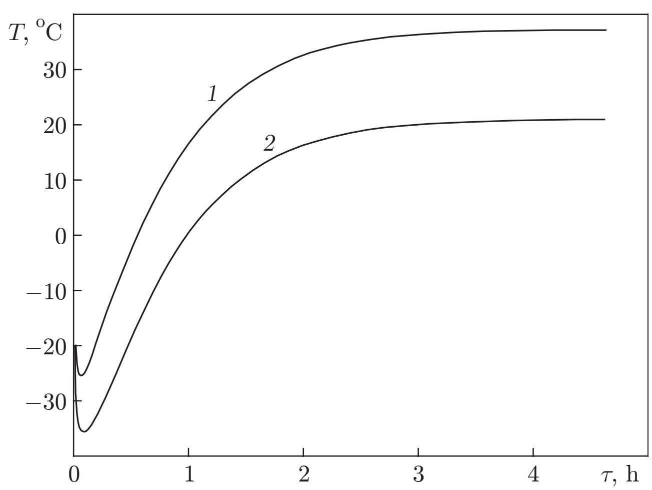

Fig. 2. Time variation of temperature in the lower part cylinder (1), and upper part of the cylinder (2).

Figure 2 shows the curves of temperature versus time at the points $[r = b,\ z = c + (d - c)/5]$ and $[r = b,\ z = e-(e-d)/5]$ in the upper and lower cylinders, respectively, from the beginning of the process to the establishment of a stationary thermal state. It is seen that the temperature at these points first falls to some values and, then, rises. This is explained by closeness of these points to the surfaces $z = c$ and $z = e$ with the temperature $T_1 = T_2 = 233$ K and by the effect due to the internal heat release in the cylinders with the intensity $w_1 = w_2 = 0.29$ W/cm^3. The level to which the temperature falls additionally depends on thermophysical coefficients of the two materials.

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